Supplement DataColada[17] "No-Way interactions"
http://datacolada.org/2014/03/12/17-no-way-interactions-2

Uri Simonsohn, March 11th 2014

Note: This writeup assumes familiarity with noncentrality distributions.
You can read this Supplement (.pdf) from our *p*-curve paper as a gentle introduction.

1. General setup

We are interested in a 2x2 design leading to four means.
Study 1 considers only M_1 and M_2, prediction is $M_1>M_2$
Study 2 consider all means, prediction is $M_1-M_2>M_3-M_4$

Let n_1, n_2 be the per-cell sample sizes of studies 1 and 2, and SD represent the pooled standard deviation within a study.

2. Simple effect (Study 1)

Standard t-test for the simple effect in Study 1 is:

$$t_{simple} = \frac{(M_1-M_2)}{SE(M_1-M_2)}$$

It will be useful to express t_{simple} in terms of SD. Let's begin doing so with $SE(M_1-M_2)$. Let's square SE to use the VAR(A-B)=VAR(A)+VAR(B) formula:

$$SE(M_1-M_2)^2 = SE(M_1)^2+SE(M_2)^2$$
$$=2*SE(M)^2$$

Take the square-root:

$$SE(M_1-M_2) = \sqrt{2}SE(M) = \sqrt{2}\,\frac{SD}{\sqrt{n_1}} = \sqrt{\frac{2}{n_1}}SD$$

Leading us to the well-known t-test formula for two-sample, same n, same variance:

$$t_{simple} = \frac{(M_1-M_2)}{\sqrt{\frac{2}{n}}SD}$$

What's interesting for us is that t_{simple}, evaluated at population values (μ:means, σ:SD) is the noncentrality parameter (NCP) for Study 1.

$$\textbf{(1) NCP}_{\textbf{simple}} = \frac{(\mu_1-\mu_2)}{\sqrt{\frac{2}{n_1}}\sigma}$$

Expressed in standardized effect size, think of it as the underlying Cohen-d, we do $\delta=(\mu_1-\mu_2)/\sigma$ and get the more common expression:

(1') $NCP_{simple}=\delta\sqrt{\frac{n_1}{2}}$

3. Interaction (Study 2) - Full elimination of the effect

For 2x2 designs, textbooks and software like gPower would turn to ANOVA, and hence the F-distribution and a different (though equivalent) metric of effect size and noncentrality parameter, but for our purposes sticking with the t makes things much more intuitive . Mathematically it is all exactly the same whichever approach we take (recall that the $F(1,df)=t(df)^2$).

Anyway. The t-test for the interaction is

$t_{inter} = \frac{(M_1-M_2)-(M_3-M_4)}{SE[(M_1-M_2)-(M_3-M_4)]}$

Proceeding analogously to above
$SE[(M_1\text{-}M_2)\text{-}(M_3\text{-}M_4)]^2=SE(M_1)^2+SE(M_2)^2 + SE(M_3)^2+SE(M_4)^2$

$=4*SE(M)^2$

Take the squared root:
$SE[(M_1\text{-}M_2)\text{-}(M_3\text{-}M_4)] = \sqrt{4}SE(M) =\sqrt{4}\,\frac{SD}{\sqrt{n_2}} = \sqrt{\frac{4}{n_2}}SD$

And hence:

$t_{inter} = \frac{(M_1-M_2)-(M_3-M_4)}{\sqrt{\frac{4}{n_2}}SD}$

The difference between t_{simple} and t_{inter} is that the latter has $(M_3\text{-}M_4)$ in the numerator, and has a denominator with 4 rather than 2 in the squared-root (for there are four means we are comparing, not two, and hence four variances we are adding up, not two).

Evaluating at population values t_{inter} becomes the noncentrality parameter for the test of the interaction in Study 2.

(2) NCP_{inter}= $\frac{(\mu_1-\mu_2)-(\mu_3-\mu_4)}{\sqrt{\frac{4}{n_2}}\sigma}$

If the effect is ***fully*** attenuated, as in eliminated, $\mu_3=\mu_4$ and hence

(2') **NCP_{inter}=** $\frac{(\mu_1-\mu_2)}{\sqrt{\frac{4}{n_2}}\sigma}$

(2'') $\mathbf{NCP_{inter}}$ $=\delta\sqrt{\frac{n_2}{4}}$

We can now find n_2, as a function of n_1, that would achieve the same NCP for Study 2 and Study 1.

(3) $NCP_{inter}=NCP_{simple}$

Substituting for (2'') and (1') into (3) we get:

(3') $\delta\sqrt{\frac{n_2}{4}} = \delta\sqrt{\frac{n_1}{2}}$

Which is true if $n_2=2*n_1$

In words, for both tests to have the same NCP, $n_2=2*n_1$.
Power will be almost identical if the NCPs are, see point 5 below.

4. Interaction (Study 2) - Partial elimination of the effect

What if the effect is not fully eliminated by the moderator?
Let's go back to equation (2)

(2) $\mathbf{NCP_{inter}}$= $\frac{(\mu_1-\mu_2)-(\mu_3-\mu_4)}{\sqrt{\frac{4}{n}}\sigma}$

Imagine an effect size of δ is attenuated in 70% to .3*δ in Study 2.
(such that $\frac{(\mu_1-\mu_2)}{\sigma}=\delta$, $\frac{(\mu_3-\mu_4)}{\sigma}=.3*\delta$):

$NCP_{simple} = \delta\sqrt{\frac{n_1}{2}}$
and
$NCP_{inter} = .7\,\delta\sqrt{\frac{n_2}{4}}$

For equal NCPs we now need:
$.7\,\delta\sqrt{\frac{n_2}{4}} = \delta\sqrt{\frac{n_1}{2}}$

$n_2=\frac{1}{.7^2}*2*n_1=4.08*n_1$

If an effect is attenuated 70%, Study 2 needs ***four*** times as many subjects, per cell, as Study 1.

5. Degrees of freedom issue.
The calculations above have equalized NCP. The same NCP leads to slightly higher power as the d.f. of the test increase, which of course happens as we increase sample size. This technical point lacks practical consequences.

For example. Say Study 1 had n=20 and was powered to 80% (so δ=.9096)
If Study 2 had n=40 and the effect is fully eliminated with the moderator, power is 81% for the interaction (we were aiming for 80%). See R code below for calculations.

R Syntax:
```
library(pwr)
d=pwr.t.test(n=20,power=.8)$d                    #Effect size that would give n=20 80% power for Study 1
ncp=sqrt(20/2)*d                                 #Implied non-centrality parameter
tc=qt(.975,df=78)                                #Critical t-value for df=78
1-pt(tc,df=78,ncp=sqrt(10)*d)+pt(-tc,df=78,ncp=sqrt(10)*d)    #Actual power for interaction with 2*n
```